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Quantum discrete Dubrovin equations

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Abstract
The discrete equations of motion for the quantum mappings of KdV type are given in terms of the Sklyanin variables (which are also known as quantum separated variables). Both temporal (discrete-time) evolutions and spatial (along the lattice at a constant time-level) evolutions are considered. In the classical limit, the temporal equations reduce to the (classical) discrete Dubrovin equations as given in a previous publication (Nijhoff F W 2000 *Chaos Solitons Fractals* **11** 19–28). The reconstruction of the original dynamical variables in terms of the Sklyanin variables is also achieved.

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1. Introduction

The quantization of discrete-time systems is an outstanding problem within the wider context of the quantum theory of dynamical systems. From the perspective of quantum chaos theory (quantum chaology), quantum mappings have been studied by Berry *et al* [6]. Quantum mechanical systems evolving in discrete-time have also been considered by Bender *et al* [4, 5], where the emphasis was on the application of finite-element methods. Both of these approaches deal with discrete-time systems that, in the classical limit, generically exhibit chaos.

Integrable classical mappings, as opposed to mappings that exhibit chaos, have been systematically constructed and studied in recent years (see, for example, [11, 25, 26, 32–34]). A specific set of examples are the so-called mappings of KdV type [8, 24], which were classically constructed from periodic initial value problems on the lattice KdV [23] partial difference equation. They are the principal model of interest in this paper. Integrable mappings are highly nontrivial; however, there is a great deal of understanding of them, both on the level of their solvability and on their classification.

Historically the quantization of integrable models formed many of the paradigms of quantum theory. Furthermore, the inherent discreteness of quantum theory points to a specific role that discrete, and hence integrable discrete, systems may play in the further development of the theory. Quantum integrability on the discrete spacetime lattice has been considered in a number of papers (mostly from the perspective of R-matrix theory), for example, [7, 10, 20, 21, 27, 35]. Here the discrete aspect is not seen as some kind of approximation but, rather, it is postulated as the underlying structure of spacetime from the very start. In the present paper, adopting the same point of view, we will be mainly concerned with the approach initiated in [7, 20, 21], where a non-ultralocal Yang–Baxter (R-matrix) structure appropriate for obtaining an 'integrable quantization' of the mappings of KdV type was given (in the continuous-time setting such a non-ultralocal Yang–Baxter structure had been previously used in connection with the quantum Toda theory [2]).

Quantizing a discrete-time system is essentially different to the conventional quantization procedure of Hamiltonian systems. In the continuous-time setting it is the Hamiltonian (or commuting family of Hamiltonians) that stand central to the theory, the spectrum and eigenfunctions of which are the main objects to be computed. The equations of motion in terms of the canonical operators (in the Heisenberg picture), or the evolution of the states (in the Schrödinger picture), only play a subsidiary role. In the discrete-time setting it is no longer the Hamiltonian(s) that define the model, but the equations of motion exclusively. This draws us away from the conventional schemes of quantization, and leads us to investigate more closely the quantum equations of motion. Following the historical imperative once again, integrability offers a leading principle to develop our understanding of quantum mechanics, this time in the discrete regime.

By an integrable quantum mapping we mean an automorphism of the quantum algebra under consideration, which, furthermore, possesses a 'sufficient' set of exact commuting invariant functions on the algebra. (We will make this definition more precise later.) This is the quantum counterpart of the classical integrability of mappings in the sense of Liouville–Arnol'd–Veselov [33]. On the classical level, mappings of this type typically involve rational expressions exhibiting singularities that imply that the time evolution cannot be globally defined. The characterization of integrable maps through their singularity structure is a focus point of current investigation (see, for example, [12, 28]). In the more restrictive context of the mappings of KdV type, a natural resolution is to describe the mapping in terms of variables that live on the Riemann surface associated with the underlying spectral curve. The explicit description of the dynamics on the Riemann surface was achieved in [17] (see also [9, 22]) leading to the so-called *discrete Dubrovin equations*.

With this insight, one may take the point of view that the proper quantization procedure for this discrete-time system is to write the equations of motion in terms of the quantum analogue of these variables. These variables are Sklyanin's variables (which are also known as quantum separated variables). Following the correspondence with the classical case, the equations of motion in terms of the Sklyanin variables are called *quantum discrete Dubrovin equations*. (It should be noted that the Sklyanin variables are taking an increasingly primary role in the field of integrable systems. They have played a fundamental part in various recent publications, with motivations different to that of this work, such as [1, 3, 31].)

The outline of the paper is as follows. The non-ultralocal Yang–Baxter structure of [21] is recapitulated in section 2, in such a way as to bring the features required for the derivation of the quantum discrete Dubrovin equations to the fore. More specific information pertaining to the mappings of KdV type is given in section 3. In section 4 the Sklyanin algebra is set up, this paves the way for the derivation of the quantum discrete Dubrovin equations in section 5. (Equations (5.14) and (5.20) give a temporal, that is discrete-time, evolution and, hence, are called *temporal quantum discrete Dubrovin equations*. Equations (5.29) and (5.31) give a spatial evolution and, hence, are called *spatial quantum discrete Dubrovin equations*.) The reconstruction of the original dynamical variables (of the mappings of KdV type) in terms of the Sklyanin variables is also addressed in section 5. The well-defined evolution arising from the quantum discrete Dubrovin equations and the reconstruction is illustrated in the one and two degrees of freedom situations in section 6. We remain very formal and algebraic throughout this paper, principally concentrating on the derivation of the quantum discrete Dubrovin equations.

2. Non-ultralocal Yang–Baxter structure

The non-ultralocal Yang–Baxter structure for the class of discrete-time systems to which the mappings of KdV type belong was given in [21]. The convention of that paper (the standard convention) will be employed in this section, appendix A, and appendix B. This convention includes that the subscripts 1, 2, ... denote factors in a matricial tensor product, and the same subscripts distinguish the associated spectral parameters. Care must be taken not to confuse these subscripts with those that correspond to the grading of the monodromy matrix or the subscripts which identify different dynamical variables. These are, however, all perfectly clear within their context.

The only nontrivial commutation relations between the operators $L_n(\lambda)$ are those on the same and nearest-neighbour sites, namely as follows,

$$R^{+}_{12}L_{n,1}L_{n,2} = L_{n,2}L_{n,1}R^{-}_{12}, \tag{2.1a}$$

$$L_{n+1,1}S^{+}_{12}L_{n,2} = L_{n,2}L_{n+1,1}, \tag{2.1b}$$

$$L_{n,1}L_{m,2} = L_{m,2}L_{n,1} \qquad |n-m| \geqslant 2, \tag{2.1c}$$

where $L_n(\lambda)$ is the L operator at the nth site and $L_{n,j}$ denotes $L_n(\lambda)$ acting nontrivially only on the jth factor of the tensor product,

$$L_{n,j} := \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \underbrace{L_n(\lambda_j)}_{j\text{th place}} \otimes \cdots \otimes \mathbf{1}.$$

The operators $R^{\pm}_{jk} := R^{\pm}_{jk}(\lambda_j, \lambda_k)$ act nontrivially only on the jth and kth factors of the tensor product. As was shown in [21], equations (2.1a) to (2.1c) lead to

$$R^{+}_{12}T^{\pm}_{n,1}T^{\pm}_{n,2} = T^{\pm}_{n,2}T^{\pm}_{n,1}R^{-}_{12}, \tag{2.2a}$$

$$T^{+}_{n,1}S^{+}_{12}T^{-}_{n,2} = T^{-}_{n,2}S^{-}_{12}T^{+}_{n,1}, \tag{2.2b}$$

where $S^{+}_{12} = S^{-}_{21}$,

$$T^{+}_{n}(\lambda) := \overleftarrow{\prod_{j=n+1}^{P}} L_j(\lambda) \qquad T^{-}_{n}(\lambda) := \overleftarrow{\prod_{j=1}^{n}} L_j(\lambda), \tag{2.3}$$

($P \in \mathbb{N}$ denotes the 'spatial' periodicity of the model, $1 \leqslant n \leqslant P-1$) and also lead to

$$R^{+}_{12}T_1S^{+}_{12}T_2 = T_2S^{-}_{12}T_1R^{-}_{12}, \tag{2.4}$$

where T_1 denotes the monodromy matrix, $T(\lambda)$, acting nontrivially only on the first factor of the tensor product; the monodromy matrix,

$$T(\lambda) := \overleftarrow{\prod_{n=1}^{P}} L_n(\lambda). \tag{2.5}$$

(The convention for the ordered product in (2.3) and (2.5) is such that the matrices L_n are ordered from right to left with increasing label n.) In the $P = 2$ case equation (2.2b) replaces (2.1b). The compatibility relations of equations (2.1a) to (2.1c) lead to the following consistency conditions on $R^{\pm}$ and $S^{\pm}$:

$$R^{\pm}_{12}R^{\pm}_{13}R^{\pm}_{23} = R^{\pm}_{23}R^{\pm}_{13}R^{\pm}_{12}, \qquad (2.6a)$$

$$R^{\pm}_{23}S^{\pm}_{12}S^{\pm}_{13} = S^{\pm}_{13}S^{\pm}_{12}R^{\pm}_{23}. \qquad (2.6b)$$

Equation (2.6a) is the quantum Yang–Baxter equation for $R^{\pm}$, which is coupled with $S^{\pm}$ by equation (2.6b). It is also assumed that S^{-}_{12} and S^{+}_{12} are invertible. In order to establish that the structure given by the above commutation relations allows for suitable commutation relations for the monodromy matrix we need to impose in addition to (2.6a) and (2.6b) that

$$R^{\pm}_{12}S^{\pm}_{12} = S^{\mp}_{12}R^{\mp}_{12}. \qquad (2.7)$$

Integrable mappings follow from a discrete-time Zakharov–Shabat system of the form

$$\widetilde{L}_n(\lambda)M_n(\lambda) = M_{n+1}(\lambda)L_n(\lambda), \qquad (2.8)$$

where the tilde ,$\widetilde{\ }$, denotes a time update and $M_n(\lambda)$ is the discrete-time evolution operator at the site n ($\widetilde{M}_n(\lambda)$ would denote the discrete-time evolution operator at the site n at the next time level). The M (or temporal) part of the extended Yang–Baxter structure, as given in [7, 20, 21], allows one to derive the invariants of the discrete-time evolution. It also allows one to show that the Yang–Baxter relation (2.4) is preserved throughout the (discrete) time evolution. The only extra relations from the M part of the extended Yang–Baxter structure required for these proofs are

$$R^{+}_{12}M_{n,1}M_{n,2} = M_{n,2}M_{n,1}R^{-}_{12} \qquad (2.9)$$

and

$$T_1M^{-1}_{1,1}S^{+}_{12}M_{1,2} = M_{1,2}S^{-}_{12}T_1M^{-1}_{1,1}. \qquad (2.10)$$

The proofs are given in appendix A.

In the quantum discrete-time setting the commuting family of invariants (i.e., the invariants of the discrete-time evolution) are given by expanding

$$\tau(\lambda) = \mathrm{tr}(K(\lambda)T(\lambda)) \qquad (2.11)$$

in powers of the spectral parameter, λ. The proof follows by taking the trace over both spaces of the tensor product of $P_{12}K_1K_2$ multiplying equation (2.10) from the left (where P_{12} is the permutation operator). (The details can be found in appendix A.) The result is that

$$K_2 = \mathrm{tr}_1\left\{P^{t_1}_{12}\left[\left({}^{t_1}S^{+}_{12}\right)^{-1}\right]\right\}, \qquad (2.12)$$

where the left superscript t_1 denotes the matrix transpose in the first factor of the matricial tensor product. Observe that the commutation relation for the L_n operators, equation (2.1a), is of the same form as that for the M_n operators, equation (2.9), and it follows immediately from equation (2.2b) that

$$T_1L^{-1}_{1,1}S^{+}_{12}L_{1,2} = L_{1,2}S^{-}_{12}T_1L^{-1}_{1,1}, \qquad (2.13)$$

which is of the same form as equation (2.10). Hence it follows, in an exactly analogous fashion to the temporal evolution, that there is a 'spatial' evolution which preserves the Yang–Baxter relation (2.4) and the family of invariants (2.11). This is proved (in both the spatial and temporal case) in appendix A. The spatial evolution is denoted by the hat ,$\widehat{\ }$, hence

$$\widehat{T}(\lambda) = L_1(\lambda)T(\lambda)L_1(\lambda)^{-1}. \qquad (2.14)$$

For later purposes it is now assumed that R^-_{12} is proportional to a rank-one projector for a particular relative value of the spectral parameters λ_1 and λ_2. This occurs for a number of quantum models [13]. From equations (2.4) and (2.7),

$$R^-_{12}S^-_{12}T_1S^+_{12}T_2 = S^+_{12}T_2S^-_{12}T_1R^-_{12}. \tag{2.15}$$

Assuming the particular relative value of λ_1 and λ_2 such that R^-_{12} is proportional to a rank-one projector, the quantum determinant [14, 15] is denoted by Δ, where

$$R^-_{12}\Delta = R^-_{12}S^-_{12}T_1S^+_{12}T_2. \tag{2.16}$$

From equations (2.1*a*) and (2.7)

$$R^-_{12}S^-_{12}L_{n,1}L_{n,2} = S^+_{12}L_{n,2}L_{n,1}R^-_{12}. \tag{2.17}$$

Maintaining the same particular relative value of λ_1 and λ_2, the local quantum determinant is denoted by $\mathrm{Qet}(L_n)$, where

$$R^-_{12}\,\mathrm{Qet}(L_n) = R^-_{12}S^-_{12}L_{n,1}L_{n,2}. \tag{2.18}$$

In appendix B it is shown that the quantum determinant factorizes in terms of the local quantum determinants as

$$\Delta = \overleftarrow{\prod_{n=1}^{P}}\,\mathrm{Qet}(L_n). \tag{2.19}$$

In section 3 the quantum mappings of KdV type are considered. The quantum determinant and the local quantum determinants are central elements of the algebra for this model. Indeed, the quantum determinant will play a central role, in both the mathematical and conventional English sense, throughout the rest of this paper.

3. Quantum mappings of KdV type

In operator form the quantum mappings of KdV type, as introduced in [20, 21], are

$$\widetilde{v}_{2j-1} = v_{2j} \qquad \widetilde{v}_{2j} = v_{2j+1} - a{v_{2j}}^{-1} + a{v_{2j+2}}^{-1} \qquad (j = 1, \ldots, P), \tag{3.1}$$

with imposed periodicity condition $v_{i+2P} = v_i$, $P \in \mathbb{N}$. The dynamical variables, v_n, are Hermitian operators, a is a real number parameter. In the notation of [20] the commutation relations of the dynamical variables read

$$[v_j, v_{j'}] = h(\delta_{j,j'+1} - \delta_{j+1,j'}), \tag{3.2}$$

($h = -\mathrm{i}\hbar$, where $\hbar$ is Planck's constant divided by 2π). The periodic initial value problem, from which the mapping arose, imposes on the mapping (3.1) the Casimirs

$$\sum_{j=1}^{P} v_{2j} = \sum_{j=1}^{P} v_{2j-1} =: \nu, \tag{3.3}$$

in such a way as to leave the value of these Casimir operators as a free parameter (it can easily be seen from the commutation relation, equation (3.2), that this is a Casimir) hence, in the classical limit, we obtain what could be called a $(P-1)$-dimensional configuration space generalization of the McMillan mapping [16]. We assume that $\nu \neq 0$.

We need to point out that, at this stage, we are only concerned with the algebraic structures behind the integrability of the quantum discrete-time systems. Hence, as far as this paper is concerned, we will deal with operators, such as the $\{v_k\} := \{v_k\}_{k=1,\ldots,2P}$, on a strictly formal level (in the spirit of related work [10]). This involves, for instance, assumptions on the invertibility of the operators, disregarding, for the time being, questions concerning the domains of the Hilbert spaces on which they act (we aim to return to this latter issue in subsequent publications).

The Lax description of the mappings of KdV type is as follows:

$$L_j = V_{2j}V_{2j-1} \qquad V_i = \begin{pmatrix} v_i & 1 \\ \lambda_i & 0 \end{pmatrix} \tag{3.4}$$

and $\lambda_{2j} = \lambda$, $\lambda_{2j+1} = \lambda + a$. The associated monodromy matrix, $T(\lambda)$, is obtained by gluing the elementary translation matrices L_j along a line connecting the sites 1 and $P + 1$ over one period P, namely

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} := \overleftarrow{\prod_{n=1}^{P}} L_n(\lambda). \tag{3.5}$$

The monodromy matrix has a natural grading in terms of the spectral parameter, λ,

$$T(\lambda) = \begin{pmatrix} \lambda^P + \lambda^{P-1}A_{P-1} + \cdots + A_0 & \lambda^{P-1}B_{P-1} + \lambda^{P-2}B_{P-2} + \cdots + B_0 \\ \lambda^P C_P + \lambda^{P-1}C_{P-1} + \cdots + \lambda C_1 & \lambda^P + \lambda^{P-1}D_{P-1} + \cdots + \lambda D_1 \end{pmatrix}. \tag{3.6}$$

Observe that $A(\lambda)$ and $D(\lambda)$ are both monic polynomials in λ. The time evolution is given by

$$\widetilde{T}(\lambda) = M(\lambda)T(\lambda)M(\lambda)^{-1} \qquad M_n = \begin{pmatrix} w_n & 1 \\ \lambda & 0 \end{pmatrix}, \tag{3.7}$$

where M is M_1, the discrete-time evolution operator at lattice site 1. More explicitly, bearing in mind that we are dealing with noncommuting operators, this gives us

$$\widetilde{T}(\lambda) = \begin{pmatrix} wB + D & \frac{1}{\lambda}(wA + C - wBw - Dw) \\ \lambda B & A - Bw \end{pmatrix}, \tag{3.8}$$

where $w := w_1$. As well as the mapping (3.1), the Zakharov–Shabat condition (2.8) reveals that

$$w_n = v_{2n-1} + \frac{a}{v_{2n}}. \tag{3.9}$$

For the mappings of KdV type, the realization of the R and S matrices, which are solutions of the compatibility relations (2.6a) and (2.6b) under the condition (2.7), is as follows:

$$\begin{aligned} R^+_{12} &= R^-_{12} - S^+_{12} + S^-_{12} \\ R^-_{12} &= \mathbf{1} \otimes \mathbf{1} + h\frac{P_{12}}{\lambda_1 - \lambda_2} \\ S^+_{12} &= \mathbf{1} \otimes \mathbf{1} - \frac{h}{\lambda_2} F \otimes E \qquad S^-_{12} = S^+_{21}, \end{aligned} \tag{3.10}$$

where the permutation operator P_{12} and the matrices E and F are given by

$$P_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.11}$$

The realization (3.10) is assumed throughout the rest of the paper.

Classically equation (3.7) gives us that the trace of the monodromy matrix is invariant under the discrete-time evolution. This argument no longer holds in the quantum case, as some of the matrix entries consist of noncommuting operators. As stated in the previous section, within the quantum case the invariants are given by expanding equation (2.11) in powers of the spectral parameter, λ. For mappings of KdV type equation (2.12) gives

$$K(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + \frac{h}{\lambda} \end{pmatrix} \tag{3.12}$$

thus

$$\tau(\lambda) = A(\lambda) + \left(1 + \frac{h}{\lambda}\right) D(\lambda). \tag{3.13}$$

Equation (3.10) shows that R_{12}^{-} is the fully antisymmetric projector, $\mathbf{1} - P_{12}$, when $\lambda_2 = \lambda_1 + h$. Therefore, with $\lambda_1 = \lambda - \frac{h}{2} =: \lambda_-$ and $\lambda_2 = \lambda + \frac{h}{2} =: \lambda_+$, equation (2.16) gives the quantum determinant for this model. It may be expressed as

$$\Delta(\lambda) = \frac{\lambda_+}{\lambda_-} D(\lambda_-) A(\lambda_+) - B(\lambda_-) C(\lambda_+). \tag{3.14}$$

There are, of course, other equivalent expressions which can be obtained using the algebra (2.4), for instance

$$\Delta(\lambda) = \frac{\lambda_+}{\lambda_-} A(\lambda_+) D(\lambda_-) - \frac{\lambda_+}{\lambda_-} B(\lambda_+) C(\lambda_-). \tag{3.15}$$

Similarly, if we write

$$L_n(\lambda) = \begin{pmatrix} a_n(\lambda) & b_n(\lambda) \\ c_n(\lambda) & d_n(\lambda) \end{pmatrix}, \tag{3.16}$$

then it can easily be shown using (2.18) that the quantum determinant of the algebra (2.1a) can be written as

$$\mathrm{Qet}(L_n(\lambda)) = \frac{\lambda_+}{\lambda_-} d_n(\lambda_-) a_n(\lambda_+) - b_n(\lambda_-) c_n(\lambda_+). \tag{3.17}$$

Hence, from (3.4), $\mathrm{Qet}(L_n(\lambda)) = \lambda_+(\lambda_+ + a)$ and, therefore, from (2.19),

$$\Delta(\lambda) = \lambda_+^P (\lambda_+ + a)^P, \tag{3.18}$$

which manifestly belongs to the centre of the algebra.

4. Sklyanin variables

Following Sklyanin [29] the operator zeros of $B(\lambda), x_n$, provide the separated canonical variables. By 'operator zeros' it is meant that

$$B(\lambda) = B_{P-1} \prod_{n=1}^{P-1} (\lambda - x_n), \tag{4.1}$$

where the $\{x_i\} := \{x_i\}_{i=1\ldots P-1}$ mutually commute. (In appendix C it is shown that B_{P-1} is equal to the Casimir (3.3). We assume the mutual commutativity of the $\{x_i\}$, this is consistent with equation (2.4).) Conjugate variables to the x_n are introduced by making the definitions

$$X_n^- = A(\lambda)\Big|_{\lambda = x_n} \qquad X_n^+ = \left(1 + \frac{h}{\lambda}\right) D(\lambda)\Big|_{\lambda = x_n} \tag{4.2}$$

where the operator ordering prescription throughout this paper is that x_n is substituted for the spectral parameter, λ, from the left, thus,

$$X_n^- = A_0 + x_n A_1 + \cdots + x_n^P, \tag{4.3a}$$

$$X_n^+ = hD_1 + x_n(D_1 + hD_2) + \cdots + x_n^P. \tag{4.3b}$$

As in [29] the full set of commutation relations between these operators follows from the Yang–Baxter structure, equation (2.4), and reads,

$$[x_m, x_n] = 0 \qquad \forall\, m, n \tag{4.4a}$$

$$X_m^{\pm} x_n = (x_n \pm h\delta_{mn}) X_m^{\pm} \qquad \forall\, m, n \tag{4.4b}$$

$$\left[X_m^{\pm}, X_n^{\pm}\right] = 0 \qquad \forall\, m, n \tag{4.4c}$$

$$\left[X_m^{-}, X_n^{+}\right] = 0 \qquad \forall\, m \neq n \tag{4.4d}$$

$$X_n^{\pm} X_n^{\mp} = \Delta\left(x_n \pm \frac{h}{2}\right) \qquad \forall\, n, \tag{4.4e}$$

where $\Delta(\lambda)$ is the quantum determinant of the model, given explicitly in equation (3.18). We also have

$$\tau(x_n) = X_n^+ + X_n^-, \tag{4.4f}$$

which leads to the linear finite-difference spectral problem known as Baxter's equation (see, for example, [29, 30]).

The derivation of the Sklyanin algebra relations (equations (4.4*a*) to (4.4*f*)) proceeds in the same way as in [29], but is slightly more involved as the initial equations from the Yang–Baxter equation are more complicated (there are extra terms due, essentially, to the non-ultralocal nature of this algebra). Remarkably, as the derivation is carried out, the extra terms vanish, leaving the Sklyanin algebra relations.

The proof of the preservation of equation (2.4), for both the temporal and spatial evolutions, is given in appendix A. As a consequence we have the following result for the Sklyanin algebra under these evolutions.

Proposition. *The Sklyanin algebra relations are preserved under both the temporal and spatial discrete evolutions.*

The (extended) Yang–Baxter structure of section 2 leads efficiently to the preservation of the Sklyanin algebra relations, as is expressed in the proposition. The Sklyanin algebra relations are the real starting point of this work. We now turn to the equations of motion in terms of the Sklyanin algebra variables.

5. Quantum discrete Dubrovin equations

The equations of motion, for the temporal and the spatial evolutions, are derived in this section. The aim is to establish these discrete evolutions in terms of the Sklyanin algebra variables; this requires the reconstruction of w and one of the original dynamical variables, $1/v_2$, in terms of the Sklyanin algebra variables. Hence the issue of the reconstruction of the original dynamical variables in terms of the Sklyanin algebra variables is also, necessarily, addressed.

In section 5.1 the invariants of both the temporal and spatial evolutions are expressed in terms of the Sklyanin algebra variables.

In section 5.2 the reconstruction of w in terms of the Sklyanin algebra variables is given. This paves the way for the derivation of equations (5.14) and (5.20). As is illustrated for the

$P = 2$ and $P = 3$ cases in section 6, these two equations, along with the preservation of the invariants, give a well-defined temporal evolution. Hence, equations (5.14) and (5.20) of section 5.2 are what we mean by the *temporal quantum discrete Dubrovin equations*.

In section 5.3 the reconstruction of $1/v_2$ in terms of the Sklyanin algebra variables is given. Along with the reconstruction of w this paves the way for the derivation of equations (5.29) and (5.31). In an exactly analogous fashion to (5.14) and (5.20) in the temporal case, equations (5.29) and (5.31) give a spatial evolution. Hence, equations (5.29) and (5.31) of section 5.3 are what we mean by the *spatial quantum discrete Dubrovin equations*. We conjecture that the spatial evolution allows for a reconstruction of all of the original dynamical variables in terms of the unshifted Sklyanin variables. This is illustrated in the $P = 3$ case in section 6, and it is not technically difficult to confirm this for the next few larger-period cases. However, the calculations quickly become very cumbersome as the period increases.

5.1. Invariants

In this section expressions are given for the invariants, which are the coefficients of the various powers of λ in equation (3.13), in terms of the Sklyanin algebra variables. From the form of the invariant in terms of entries of the monodromy matrix, (3.13), and their gradation (3.6),

$$\tau(x_n) = 2x_n^P + x_n^{P-1} I_{P-1} + x_n^{P-2} I_{P-2} + \cdots + x_n I_1 + I_0. \tag{5.1}$$

It is easily shown that I_{P-1} is a Casimir operator. Taking into account the gradation given in equation (3.6), equation (3.14) gives us that the quantum determinant

$$\Delta\left(\lambda + \frac{h}{2}\right) = \lambda^{2P} + \lambda^{2P-1}(Ph + A_{P-1} + D_{P-1} + h - B_{P-1}C_P) + \cdots.$$

However, for the mappings of KdV type we have (from equation (3.18))

$$\Delta\left(\lambda + \frac{h}{2}\right) = \lambda^{2P} + \lambda^{2P-1}(2Ph + Pa) + \cdots.$$

Therefore, using that the central elements $B_{P-1} = C_P = \nu$ (as derived in appendix C),

$$I_{P-1} := A_{P-1} + D_{P-1} + h = \nu^2 + P(a + h). \tag{5.2}$$

Now observe that

$$\begin{pmatrix} I_0 \\ I_1 \\ \vdots \\ I_{P-2} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & \dots & x_1^{P-2} \\ 1 & x_2 & \dots & x_2^{P-2} \\ \vdots & & & \vdots \\ 1 & x_{P-1} & \dots & x_{P-1}^{P-2} \end{pmatrix}^{-1} \begin{pmatrix} \tau(x_1) - 2x_1^P - x_1^{P-1} I_{P-1} \\ \tau(x_2) - 2x_2^P - x_2^{P-1} I_{P-1} \\ \vdots \\ \tau(x_{P-1}) - 2x_{P-1}^P - x_{P-1}^{P-1} I_{P-1} \end{pmatrix}. \tag{5.3}$$

All terms on the right-hand side are known (remember that $\tau(x_n) = X_n^+ + X_n^-$). Equation (5.1) may also be rewritten as

$$\begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_{P-1} \end{pmatrix} = \begin{pmatrix} x_1 & x_1^2 & \dots & x_1^{P-1} \\ x_2 & x_2^2 & \dots & x_2^{P-1} \\ \vdots & & & \vdots \\ x_{P-1} & x_{P-1}^2 & \dots & x_{P-1}^{P-1} \end{pmatrix}^{-1} \begin{pmatrix} \tau(x_1) - 2x_1^P - I_0 \\ \tau(x_2) - 2x_2^P - I_0 \\ \vdots \\ \tau(x_{P-1}) - 2x_{P-1}^P - I_0 \end{pmatrix}. \tag{5.4}$$

Note that within the VanderMonde matrices all of the entries commute.

As the quantum discrete Dubrovin equations, which are derived in the following sections, are also matricial equations, it is expedient to introduce some specialized notation. The symbol

$\mathcal{M}$ is introduced to denote the VanderMonde matrix, the symbol $\mathcal{D}$ to denote the diagonal matrix,

$$\mathcal{M} = \begin{pmatrix} x_1 & x_1^2 & \dots & x_1^{P-1} \\ x_2 & x_2^2 & \dots & x_2^{P-1} \\ \vdots & & & \vdots \\ x_{P-1} & x_{P-1}^2 & \dots & x_{P-1}^{P-1} \end{pmatrix} \qquad \mathcal{D} = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & x_{P-1} \end{pmatrix}.$$

Vectors are indicated by a bold typeface, and all are $P-1$ dimensional. The vector $\mathbf{e} = (1, 1, \dots, 1)^t$. The vector consisting of ordered entries, labelled n to $P-2+n$, for any integer n, is denoted by a bold typeface with a subscript n, for instance $\mathbf{y}_0 = (y_0, y_1, \dots, y_{P-2})^t$, $\mathbf{y}_1 = (y_1, y_2, \dots, y_{P-1})^t$. In this notation equations (5.3) and (5.4) are rewritten as

$$\mathbf{I}_0 = \mathcal{M}^{-1}\mathcal{D}\big(\mathbf{X}_1^+ + \mathbf{X}_1^- - \mathcal{D}^P 2\mathbf{e} - \mathcal{D}^{P-1} I_{P-1}\mathbf{e}\big)$$

and

$$\mathbf{I}_1 = \mathcal{M}^{-1}\big(\mathbf{X}_1^+ + \mathbf{X}_1^- - \mathcal{D}^P 2\mathbf{e} - I_0\mathbf{e}\big),$$

respectively, where $\mathbf{I}_0$ and $\mathbf{I}_1$ denote the left-hand sides of (5.3) and (5.4) respectively.

5.2. *Temporal equations*

In section 5.2.1 we give the reconstruction of w in terms of the Sklyanin algebra variables. For heuristic reasons the matricial equations are written out in full in this section. In section 5.2.2 the temporal quantum discrete Dubrovin equations are derived.

5.2.1. Reconstruction of w. Observe that, from the definitions (4.2) (and remembering that x_n are substituted for λ from the left),

$$D(x_n) := D(\lambda)\Big|_{\lambda = x_n} = \frac{x_n}{x_n + h} X_n^+.$$

(We take the casual attitude of writing $1/Y$ for the inverse of the operator Y.) Hence, still working from the definitions,

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{P-1} \end{pmatrix} = \begin{pmatrix} x_1 & x_1^2 & \dots & x_1^{P-1} \\ x_2 & x_2^2 & \dots & x_2^{P-1} \\ \vdots & & & \vdots \\ x_{P-1} & x_{P-1}^2 & \dots & x_{P-1}^{P-1} \end{pmatrix}^{-1} \begin{pmatrix} X_1^- - x_1^P - A_0 \\ X_2^- - x_2^P - A_0 \\ \vdots \\ X_{P-1}^- - x_{P-1}^P - A_0 \end{pmatrix}, \tag{5.5}$$

$$\begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_{P-1} \end{pmatrix} = \begin{pmatrix} x_1 & x_1^2 & \dots & x_1^{P-1} \\ x_2 & x_2^2 & \dots & x_2^{P-1} \\ \vdots & & & \vdots \\ x_{P-1} & x_{P-1}^2 & \dots & x_{P-1}^{P-1} \end{pmatrix}^{-1} \begin{pmatrix} \frac{x_1}{h+x_1} X_1^+ - x_1^P \\ \frac{x_2}{h+x_2} X_2^+ - x_2^P \\ \vdots \\ \frac{x_{P-1}}{h+x_{P-1}} X_{P-1}^+ - x_{P-1}^P \end{pmatrix}. \tag{5.6}$$

So, it is seen that,

$$\begin{pmatrix} A_1 - D_1 + hD_2 \\ A_2 - D_2 + hD_3 \\ \vdots \\ A_{P-1} - D_{P-1} + hD_P \end{pmatrix}$$
$$= \begin{pmatrix} x_1 & x_1^2 & \dots & x_1^{P-1} \\ x_2 & x_2^2 & \dots & x_2^{P-1} \\ \vdots & & & \vdots \\ x_{P-1} & x_{P-1}^2 & \dots & x_{P-1}^{P-1} \end{pmatrix}^{-1} \begin{pmatrix} X_1^- + \frac{h-x_1}{h+x_1} X_1^+ - I_0 \\ X_2^- + \frac{h-x_2}{h+x_2} X_2^+ - I_0 \\ \vdots \\ X_{P-1}^- + \frac{h-x_{P-1}}{h+x_{P-1}} X_{P-1}^+ - I_0 \end{pmatrix}. \tag{5.7}$$

The time-update of the monodromy matrix, equation (3.8), and a commutation relation contained within (2.10) give

$$\widetilde{A} = Bw - \frac{h}{\lambda}(A - Bw) + \left(1 + \frac{h}{\lambda}\right) D \qquad \widetilde{D} = A - Bw. \tag{5.8}$$

Substituting x_n for λ from the left, and again using the definitions (4.1) and (4.2), gives,

$$\widetilde{A}_0 + x_n \widetilde{A}_1 + \cdots + x_n^P = -\frac{h}{x_n} X_n^- + X_n^+, \tag{5.9a}$$

$$x_n \widetilde{D}_1 + x_n^2 \widetilde{D}_2 + \cdots + x_n^P = X_n^-. \tag{5.9b}$$

From here we easily obtain (using $\widetilde{A}_0 + h\widetilde{D}_1 = \widetilde{I}_0 = I_0$) that

$$\begin{pmatrix} \widetilde{A}_1 - \widetilde{D}_1 + h\widetilde{D}_2 \\ \widetilde{A}_2 - \widetilde{D}_2 + h\widetilde{D}_3 \\ \vdots \\ \widetilde{A}_{P-1} - \widetilde{D}_{P-1} + h\widetilde{D}_P \end{pmatrix} = \begin{pmatrix} x_1 & x_1^2 & \dots & x_1^{P-1} \\ x_2 & x_2^2 & \dots & x_2^{P-1} \\ \vdots & & & \vdots \\ x_{P-1} & x_{P-1}^2 & \dots & x_{P-1}^{P-1} \end{pmatrix}^{-1} \begin{pmatrix} X_1^+ - X_1^- - I_0 \\ X_2^+ - X_2^- - I_0 \\ \vdots \\ X_{P-1}^+ - X_{P-1}^- - I_0 \end{pmatrix}. \tag{5.10}$$

Equation (3.8) also gives

$$\widetilde{A}(\lambda) - \widetilde{D}(\lambda) + A(\lambda) - D(\lambda) = wB(\lambda) + B(\lambda)w.$$

A consideration of the leading term as $\lambda \to \infty$ shows that

$$\widetilde{A}_{P-1} - \widetilde{D}_{P-1} + A_{P-1} - D_{P-1} = 2B_{P-1}w, \tag{5.11}$$

as B_{P-1} is actually the Casimir (3.3) (this is shown in appendix C). Therefore

$$w = \frac{1}{2\nu}(\widetilde{A}_{P-1} - \widetilde{D}_{P-1} + A_{P-1} - D_{P-1}).$$

Expressions for $\widetilde{A}_{P-1} - \widetilde{D}_{P-1}$ and $A_{P-1} - D_{P-1}$ in terms of the Sklyanin algebra variables, $\{x_i, X_i^{\pm}\} := \{x_i, X_i^{\pm}\}_{i=1,\dots,P-1}$, follow from equations (5.7) and (5.10). To obtain an expression explicitly in terms of $\{x_i, X_i^{\pm}\}$, I_0 may be replaced by using (5.4); then Cramer's rule, along with (5.2) for the value of the central element I_{P-1}, allows one to deduce that, in terms of the Sklyanin algebra,

$$w = \frac{1}{\nu}\left[\nu^2 + Pa + (P-1)h + (-1)^P \sum_{n=1}^{P-1}\left(\prod_{\substack{i=1\\ i\neq n}}^{P-1}\frac{1}{x_i - x_n}\right)\right.$$
$$\left.\times\left(2x_n^{P-1} - \frac{1}{h+x_n}X_n^+ - \frac{1}{x_n}X_n^-\right)\right]. \tag{5.12}$$

5.2.2. Temporal quantum discrete Dubrovin equations. Equations (5.7) and (5.10) give, in the notation introduced in section 5.1,

$$\widetilde{\mathcal{M}}^{-1}\big(\widetilde{\mathbf{X}}_1^- + (h\mathbf{1}+\widetilde{\mathcal{D}})^{-1}(h\mathbf{1}-\widetilde{\mathcal{D}})\widetilde{\mathbf{X}}_1^+ - I_0\,\mathbf{e}\big) = \mathcal{M}^{-1}\big(\mathbf{X}_1^+ - \mathbf{X}_1^- - I_0\,\mathbf{e}\big). \tag{5.13}$$

This constitutes the (temporal part of the) quantum discrete Dubrovin equations, as given classically in [17]. Note that the time-evolved variables still obey the relations (4.4*a*) to (4.4*f*) (as stated in the proposition of section 4). Along with the time-update invariance of (5.4), (5.13) leads to

$$\widetilde{\mathcal{M}}^{-1}\big(\widetilde{\mathcal{D}}^P\mathbf{e} - (h\mathbf{1}+\widetilde{\mathcal{D}})^{-1}\widetilde{\mathcal{D}}\widetilde{\mathbf{X}}_1^+\big) = \mathcal{M}^{-1}(\mathcal{D}^P\mathbf{e} - \mathbf{X}_1^-). \tag{5.14}$$

The equations for the elementary symmetric polynomials in $\{\tilde{x}_i\}$ in terms of $\{x_i, X_i^\pm\}$ will now be obtained. A consideration of equation (4.1) shows that these follow immediately from the coefficients of different powers of λ in $\widetilde{B}(\lambda)$, that is, from $\{\widetilde{B}_0, \ldots, \widetilde{B}_{P-2}\}$. Equation (3.8) and a commutation relation contained within (2.10) give

$$\widetilde{B}(\lambda) = \frac{1}{\lambda}\left[Aw - Bw^2 + \left(1+\frac{h}{\lambda}\right)(C - Dw)\right]. \tag{5.15}$$

Hence, to obtain the elementary symmetric polynomials in $\{\tilde{x}_i\}$ in terms of $\{x_i, X_i^\pm\}$, $C(\lambda)$ must be expressed in terms of $\{x_i, X_i^\pm\}$. From equations (3.14) and (3.18) for the quantum determinant,

$$B(\lambda-h)C(\lambda) = \frac{\lambda}{\lambda-h}D(\lambda-h)A(\lambda) - \lambda^P(\lambda+a)^P. \tag{5.16}$$

Consider the first term on the right-hand side,

$$\frac{\lambda}{\lambda-h}D(\lambda-h)A(\lambda) = \sum_{j=0}^{2P-1}\lambda^{j+1}\sum_{\substack{k=0\\ k\geqslant j-P\\ k\leqslant P-1}}^{j}\left[\sum_{l=k}^{P-1}\binom{l}{k}(-h)^{l-k}D_{l+1}\right]A_{j-k}, \tag{5.17}$$

where $D_P = 1$ and $\binom{l}{k}$ denotes the binomial coefficient, $\binom{l}{k} = l!/[(l-k)!k!]$. Whence, substituting x_n for λ from the left,

$$C(x_n) = \frac{1}{B_{P-1}}\left(\prod_{l=1}^{P-1}\frac{1}{x_n - x_l - h}\right)$$
$$\times\left\{\sum_{j=0}^{2P-1}x_n^{j+1}\sum_{\substack{k=0\\ k\geqslant j-P\\ k\leqslant P-1}}^{j}\left[\sum_{l=k}^{P-1}\binom{l}{k}(-h)^{l-k}D_{l+1}\right]A_{j-k} - x_n^P(x_n+a)^P\right\}. \tag{5.18}$$

A consideration of the λ^0 term of $\widetilde{D}(\lambda)$ in equation (5.8) reveals that $A_0 = B_0 w$. As w is given in terms of the Sklyanin algebra, $\{x_i, X_i^\pm\}$, in equation (5.12), we have, along with equations (5.5) and (5.6), all of $\{A_n\}$ and $\{D_n\}$ in terms of $\{x_i, X_i^{\pm}\}$. Therefore equation (5.18) gives $C(x_n)$ entirely in terms of the Sklyanin algebra. Consider now equation (5.15) with x_n substituted for λ from the left, this defines

$$\widetilde{B}(x_n) = \frac{1}{x_n}\left[\left(X_n^- - X_n^+\right)w + \left(1 + \frac{h}{x_n}\right)C(x_n)\right]. \tag{5.19}$$

The right-hand side of (5.19) may be expressed entirely in terms of $\{x_i, X_i^{\pm}\}$, for brevity denote the right-hand side, strictly in terms of $\{x_i, X_i^{\pm}\}$, by $\widetilde{\mathcal{B}}_n$. Therefore, with $\widetilde{\boldsymbol{\mathcal{B}}}_1 = (\widetilde{\mathcal{B}}_1, \widetilde{\mathcal{B}}_2, \ldots, \widetilde{\mathcal{B}}_{P-1})^t$, we have the following expression,

$$\widetilde{\mathbf{B}}_0 = \mathcal{M}^{-1}\mathcal{D}(\widetilde{\boldsymbol{\mathcal{B}}}_1 - \mathcal{D}^{P-1}B_{P-1}\mathbf{e}), \tag{5.20}$$

which gives the elementary symmetric polynomials in $\{\tilde{x}_i\}$ in terms of $\{x_i, X_i^{\pm}\}$.

Equations (5.14) (or (5.13)) and (5.20) constitute the temporal quantum discrete Dubrovin equations.

5.3. Spatial equations

In this section we consider an evolution along the lattice at a constant time level. Specifically,

$$T = L_P L_{P-1} L_{P-2} \ldots L_2 L_1 \mapsto \widehat{T} = L_1 L_P L_{P-1} \ldots L_3 L_2 \mapsto \cdots. \tag{5.21}$$

(The hat, $\widehat{\ }$, is used to denote the evolution along the lattice at a constant time-level.) In the temporal case the reconstruction of w is required for the quantum discrete Dubrovin equations. The spatial equations also require the reconstruction of $1/v_2$, this is achieved in section 5.3.1. Together with equation (3.9), this also gives the reconstruction of v_1. If one requires a reconstruction of the original dynamical variables, $\{v_k\}$, in terms of the Sklyanin algebra set up at each dynamical variable's particular lattice site (that is, v_1 and v_2 in terms of $\{x_i, X_i^{\pm}\}$, v_3 and v_4 in terms of $\{\hat{x}_i, \widehat{X}_i^{\pm}\}$, etc) then the reconstruction is complete. However, we conjecture that the spatial quantum discrete Dubrovin equations, as derived in section 5.3.2, allow for a reconstruction of all of the original dynamical variables in terms of the unshifted Sklyanin variables $\{x_i, X_i^{\pm}\}$.

5.3.1. Reconstruction of $\frac{1}{v_2}$. In appendix C it is shown that B_{P-1} is equal to the Casimir (3.3), it is obvious that $\widehat{B}_{P-1}$ also is (as it is still the sum over all $\{v_{2j}\}$). Now, from the definitions of the 'conjugate variables' in section 4,

$$\mathbf{A}_1 - \mathbf{D}_1 + h\mathbf{D}_2 \pm 2\mathbf{B}_0\frac{1}{v_2} = \mathcal{M}^{-1}\left(\mathbf{X}_1^- + (h\mathbf{1} + \mathcal{D})^{-1}(h\mathbf{1} - \mathcal{D})\mathbf{X}_1^+ - I_0\,\mathbf{e} \mp \mathcal{D}^P 2B_{P-1}\frac{1}{v_2}\,\mathbf{e}\right). \tag{5.22}$$

Spatially updated terms follow from (5.21) (or, equivalently, (2.14)). Using the commutation relations in (2.13),

$$\widehat{A} + \frac{h}{\lambda}\widehat{D} = \frac{1}{\lambda + a}(A - Bv_1)(v_2v_1 + \lambda + a) + \left(1 + \frac{h}{\lambda}\right)\frac{1}{\lambda + a}(C - Dv_1)v_2, \tag{5.23a}$$

$$\widehat{D} = \frac{1}{\lambda + a}[-Av_2 + B(\lambda + a + v_1v_2)]v_1 + \left(1 + \frac{h}{\lambda}\right)\frac{1}{\lambda + a}[-Cv_2 + D(\lambda + a + v_1v_2)], \tag{5.23b}$$

$$\widehat{B} = \frac{1}{\lambda(\lambda+a)}[-Av_2 + B(\lambda+a+v_1v_2)](v_1v_2+h+\lambda+a)$$
$$+\left(1+\frac{h}{\lambda}\right)\frac{1}{\lambda(\lambda+a)}[-Cv_2 + D(\lambda+a+v_1v_2)]v_2. \quad (5.23c)$$

So,

$$\widehat{A} + \frac{h}{\lambda}\widehat{D} - \widehat{D} + 2\lambda\widehat{B}\frac{1}{v_2} = -\frac{\lambda+a+2h}{\lambda+a}A + \left(1+\frac{h}{\lambda}\right)D$$
$$+2\frac{(\lambda+a+h)}{\lambda+a}B(\lambda+a+v_1v_2)\frac{1}{v_2}. \quad (5.24)$$

From equation (5.24) one obtains

$$\widehat{\mathbf{A}}_1 - \widehat{\mathbf{D}}_1 + h\widehat{\mathbf{D}}_2 + 2\widehat{\mathbf{B}}_0\frac{1}{v_2}$$
$$= \mathcal{M}^{-1}\left(\mathbf{X}_1^+ - (\mathcal{D}+a\mathbf{1})^{-1}(\mathcal{D}+a\mathbf{1}+2h\mathbf{1})\mathbf{X}_1^- - I_0\,\mathbf{e} - \mathcal{D}^P 2B_{P-1}\frac{1}{v_2}\,\mathbf{e}\right). \quad (5.25)$$

It is easily shown that the leading term in equation (5.24) as $\lambda \to \infty$ gives

$$2\left(B_{P-1}w + B_{P-1}\frac{h}{v_2}\right) = \left(\widehat{A}_{P-1} - \widehat{D}_{P-1} + h + 2\widehat{B}_{P-2}\frac{1}{v_2}\right)$$
$$+\left(A_{P-1} - D_{P-1} + h - 2B_{P-2}\frac{1}{v_2}\right). \quad (5.26)$$

Equations (5.22) and (5.25) along with (5.7), (5.10) and (5.11) from the reconstruction of w, and Cramer's rule, lead to

$$\frac{1}{v_2} = \frac{1}{v}\left[1 + (-1)^{P-1}\sum_{n=1}^{P-1}\frac{1}{x_n}\left(\prod_{\substack{i=1\\i\neq n}}^{P-1}\frac{1}{x_i - x_n}\right)\frac{1}{x_n+a}X_n^-\right]. \quad (5.27)$$

Along with w, as reconstructed in equation (5.12), and equation (3.9) for w_n in terms of the dynamical variables $\{v_k\}$, equation (5.27) leads immediately to the reconstruction of v_1.

5.3.2. Spatial quantum discrete Dubrovin equations. Equations (5.22) and (5.25) give

$$\widehat{\mathcal{M}}^{-1}\left(\widehat{\mathbf{X}}_1^- + (h\mathbf{1}+\widehat{\mathcal{D}})^{-1}(h\mathbf{1}-\widehat{\mathcal{D}})\widehat{\mathbf{X}}_1^+ - I_0\,\mathbf{e} - \widehat{\mathcal{D}}^P 2B_{P-1}\frac{1}{v_2}\,\mathbf{e}\right)$$
$$= \mathcal{M}^{-1}\left(\mathbf{X}_1^+ - (\mathcal{D}+a\mathbf{1})^{-1}(\mathcal{D}+a\mathbf{1}+2h\mathbf{1})\mathbf{X}_1^- - I_0\,\mathbf{e} - \mathcal{D}^P 2B_{P-1}\frac{1}{v_2}\,\mathbf{e}\right), \quad (5.28)$$

for the spatial part of the quantum discrete Dubrovin equations. As the invariants of the time evolution are also invariants of the spatial evolution we may use them, in the form given in equation (5.4), to rewrite (5.28) as

$$\widehat{\mathcal{M}}^{-1}\left((h\mathbf{1}+\widehat{\mathcal{D}})^{-1}\widehat{\mathcal{D}}\,\widehat{\mathbf{X}}_1^+ - \widehat{\mathcal{D}}^P\left(1 - B_{P-1}\frac{1}{v_2}\right)\mathbf{e}\right)$$
$$= \mathcal{M}^{-1}\left((\mathcal{D}+a\mathbf{1})^{-1}(\mathcal{D}+a\mathbf{1}+h\mathbf{1})\mathbf{X}_1^- - \mathcal{D}^P\left(1 - B_{P-1}\frac{1}{v_2}\right)\mathbf{e}\right). \quad (5.29)$$

The expressions for the elementary symmetric polynomials in the 'spatially updated' variables, $\{\hat{x}_i\}$, in terms of $\left\{x_i, X_i^{\pm}\right\}$ are now given. Upon substituting x_n for λ from the left, equation (5.23*c*) becomes

$$\widehat{B}(x_n) = \frac{1}{x_n(x_n+a)} X_n^+ w v_2^2 - \frac{1}{x_n(x_n+a)} X_n^- v_2^2 w + \frac{1}{x_n+a} \times \left(X_n^+ - X_n^-\right) v_2 - \left(1 + \frac{h}{x_n}\right) \frac{1}{x_n(x_n+a)} C(x_n) v_2^2. \tag{5.30}$$

Note that $C(x_n)$ was expressed in terms of $\left\{x_i, X_i^{\pm}\right\}$ by equations (5.5), (5.6) and (5.18); w was reconstructed in terms of $\left\{x_i, X_i^{\pm}\right\}$ in equation (5.12); v_2 was reconstructed in terms of $\left\{x_i, X_i^{\pm}\right\}$ in equation (5.27). Therefore the right-hand side of (5.30) can be given entirely in terms of $\left\{x_i, X_i^{\pm}\right\}$. For brevity denote the right-hand side of (5.30) expressed strictly in terms of $\left\{x_i, X_i^{\pm}\right\}$ by $\widehat{\mathcal{B}}_n$. With this notation

$$\widehat{\mathbf{B}}_0 = \mathcal{M}^{-1}\mathcal{D}(\widehat{\boldsymbol{\mathcal{B}}}_1 - \mathcal{D}^{P-1} B_{P-1}\mathbf{e}), \tag{5.31}$$

giving the elementary symmetric polynomials in $\{\hat{x}_i\}$ in terms of $\left\{x_i, X_i^{\pm}\right\}$.

Equations (5.29) (or (5.28)) and (5.31) constitute the spatial quantum discrete Dubrovin equations.

6. Examples

The well-defined temporal evolution which follows from the quantum discrete Dubrovin equations is illustrated for the $P = 2$ case in section 6.1 and for the $P = 3$ case in section 6.2. The reconstruction of the original dynamical variables, $\{v_k\}$, in terms of the Sklyanin algebra variables, $\left\{x_i, X_i^{\pm}\right\}$, is also performed in the $P = 2$ and $P = 3$ cases. The reconstruction is achieved, essentially, via a 'spatial evolution' using the spatial part of the quantum discrete Dubrovin equations.

6.1. The $P = 2$ case

6.1.1. Temporal evolution. With $P = 2$, equation (5.12) gives

$$w = \frac{1}{v}\left(v^2 + 2a + h + 2x - \frac{1}{x+h}X^+ - \frac{1}{x}X^-\right). \tag{6.1}$$

Equation (5.14) gives directly that

$$\tilde{x} - \frac{1}{\tilde{x}+h}\widetilde{X}^+ = x - \frac{1}{x}X^-. \tag{6.2}$$

Along with (6.1), equation (5.20) leads to

$$\tilde{x} = \frac{1}{v} w \left(x - \frac{1}{x}X^-\right). \tag{6.3}$$

It is easily seen that

$$\left(1 + \frac{h}{x}\right)\frac{1}{x+h}X^+ = -\frac{1}{x}X^- + \frac{1}{x}I_0 + I_1 + 2x.$$

Hence we may consider the evolution given by the quantum discrete Dubrovin equations to be that of x and $\frac{1}{x+h}X^+$ (or $\frac{1}{x}X^-$) along with the preservation of the invariant, $I_0 = \widetilde{I}_0 \ldots$. Equation (5.2) gives $I_1 = v^2 + 2(a+h)$.

6.1.2. Reconstruction. Setting $P = 2$ in equation (5.27) gives

$$\frac{1}{v_2} = \frac{1}{\nu}\left(1 - \frac{1}{x(x+a)}X^-\right). \tag{6.4}$$

It follows trivially from the Casimir (3.3), $v_2 + v_4 = \nu$, that we also have v_4 in terms of $(x, X^\pm)$. Equation (3.9) along with (6.1) and (6.4) gives the reconstruction of v_1, and the Casimir (3.3), $v_1 + v_3 = \nu$, gives the reconstruction of v_3. Therefore, full reconstruction has been achieved.

6.2. The $P = 3$ case

6.2.1. Temporal evolution. For this section we define

$$\pi_1 := \frac{1}{x_2 - x_1}\left(\frac{x_2}{x_1+h}X_1^+ - \frac{x_1}{x_2+h}X_2^+\right) \qquad \pi_2 := \frac{1}{x_2 - x_1}\left(\frac{1}{x_1+h}X_1^+ - \frac{1}{x_2+h}X_2^+\right),$$
$$\phi_1 := \frac{1}{x_2 - x_1}\left(\frac{x_2}{x_1}X_1^- - \frac{x_1}{x_2}X_2^-\right) \qquad \phi_2 := \frac{1}{x_2 - x_1}\left(\frac{1}{x_1}X_1^- - \frac{1}{x_2}X_2^-\right).$$

Then, with $P = 3$, equation (5.12) gives

$$w = \frac{1}{\nu}[\nu^2 + 3a + 2h + 2(x_1 + x_2) + \pi_2 + \phi_2]. \tag{6.5}$$

Equation (5.14) gives directly that

$$\tilde{x}_1\tilde{x}_2 + \tilde{\pi}_1 = x_1x_2 + \phi_1, \tag{6.6}$$

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{\pi}_2 = x_1 + x_2 + \phi_2. \tag{6.7}$$

Along with (6.5), equation (5.20) leads to

$$\tilde{x}_1\tilde{x}_2 = \frac{1}{\nu}w(x_1x_2 + \phi_1), \tag{6.8}$$

$$\begin{aligned}\tilde{x}_1 + \tilde{x}_2 = \frac{1}{\nu}w\,(x_1 + x_2 + \phi_2) - \frac{1}{\nu^2}[&2x_1x_2 + h(x_1 + x_2) + h^2 - 3a^2 \\ &- (x_1 + x_2 + 2h)(x_1 + x_2 + 2h + 3a) + \pi_1 + \phi_1 + \pi_2\phi_2].\end{aligned} \tag{6.9}$$

It is easily seen that

$$\frac{h}{x_1x_2}\pi_1 + \pi_2 = -\phi_2 + \frac{1}{x_1x_2}I_0 - I_2 - 2(x_1 + x_2)$$

and

$$\left[1 + \frac{h(x_1 + x_2)}{x_1x_2}\right]\pi_1 - h\pi_2 = -\phi_1 + \frac{x_1 + x_2}{x_1x_2}I_0 + I_1 - 2x_1x_2.$$

Hence we may consider the evolution given by the quantum discrete Dubrovin equations to be that of the elementary symmetric polynomials x_1x_2 and $x_1 + x_2$, and π_1 and π_2 (or ϕ_1 and ϕ_2), along with the preservation of the invariants, $I_i = \widetilde{I}_i \ldots$. The form of the invariants follows from (5.2) and (5.3).

6.2.2. Reconstruction. Setting $P = 3$ in equation (5.27) gives

$$\frac{1}{v_2} = \frac{1}{\nu}\left[1 + \frac{1}{x_2 - x_1}\left(\frac{1}{x_1(x_1+a)}X_1^- - \frac{1}{x_2(x_2+a)}X_2^-\right)\right]. \tag{6.10}$$

In the $P = 3$ case, equation (5.29) of the spatial part of the quantum discrete Dubrovin equations reads

$$\begin{pmatrix} -\hat{x}_1\hat{x}_2\left(1-\frac{\nu}{v_2}\right) - \frac{1}{\hat{x}_2-\hat{x}_1}\left(\frac{\hat{x}_2}{\hat{x}_1+h}\widehat{X}_1^+ - \frac{\hat{x}_1}{\hat{x}_2+h}\widehat{X}_2^+\right) \\ (\hat{x}_1+\hat{x}_2)\left(1-\frac{\nu}{v_2}\right) + \frac{1}{\hat{x}_2-\hat{x}_1}\left(\frac{1}{\hat{x}_1+h}\widehat{X}_1^+ - \frac{1}{\hat{x}_2+h}\widehat{X}_2^+\right)\end{pmatrix}$$
$$= \begin{pmatrix} -x_1x_2\left(1-\frac{\nu}{v_2}\right) - \frac{1}{x_2-x_1}\left[\frac{x_2}{x_1}\left(1+\frac{h}{x_1+a}\right)X_1^- - \frac{x_1}{x_2}\left(1+\frac{h}{x_2+a}\right)X_2^-\right] \\ (x_1+x_2-h)\left(1-\frac{\nu}{v_2}\right) + \frac{1}{x_2-x_1}\left(\frac{1}{x_1}X_1^- - \frac{1}{x_2}X_2^-\right)\end{pmatrix}. \tag{6.11}$$

The top $-(a+h)$ bottom of (6.11) leads to

$$[-\hat{x}_1\hat{x}_2 - (a+h)(\hat{x}_1+\hat{x}_2)]\left(1-\frac{\nu}{v_2}\right) - \frac{1}{\hat{x}_2-\hat{x}_1}$$
$$\times\left(\frac{\hat{x}_2+a+h}{\hat{x}_1+h}\widehat{X}_1^+ - \frac{\hat{x}_1+a+h}{\hat{x}_2+h}\widehat{X}_2^+\right) = (a+h)^2\left(1-\frac{\nu}{v_2}\right). \tag{6.12}$$

Therefore,

$$\frac{1}{v_0} = \frac{1}{\nu}\left[1 + \frac{1}{x_2-x_1}\left(\frac{1}{(x_1+h)(x_1+a+h)}X_1^+ - \frac{1}{(x_2+h)(x_2+a+h)}X_2^+\right)\right].$$

It follows trivially from the Casimir (3.3), $v_0 + v_2 + v_4 = \nu$, that we also have v_4 in terms of $\{x_i, X_i^{\pm}\}_{i=1,2}$.

The reconstruction of $\{v_{2j+1}\}$ follows from that of w. It follows from equation (3.9) that v_1, in terms of $\{x_i, X_i^{\pm}\}_{i=1,2}$, is obtained from equations (6.5) and (6.10). Using the form of I_0 which follows from (5.3), one obtains

$$w = \frac{1}{\nu}\left[\frac{1}{x_1x_2}I_0 - h - \frac{h}{x_1x_2}\pi_1\right].$$

Now, as I_0 is invariant under spatial updates, it may be written, using the same expression from equation (5.3), at any spatial level. In simple terms, as $I_0 = \widehat{I_0} = \widehat{\widehat{I_0}} = \cdots$, the same expression for I_0 may be used with any number of hats above or below the operators. Hence,

$$\hat{w} = \frac{1}{\nu}\left[\frac{1}{\hat{x}_1\hat{x}_2}I_0 - h - \frac{h}{\hat{x}_1\hat{x}_2}\hat{\pi}_1\right]. \tag{6.13}$$

The top equation of the spatial quantum discrete Dubrovin equations, (6.11), gives

$$\frac{1}{\hat{x}_1\hat{x}_2}\hat{\pi}_1 = \left(\frac{1}{\hat{x}_1\hat{x}_2}x_1x_2 - 1\right)\left(1-\frac{\nu}{v_2}\right)$$
$$+ \frac{1}{\hat{x}_1\hat{x}_2}\frac{1}{x_2-x_1}\left[\frac{x_2}{x_1}\left(1+\frac{h}{x_1+a}\right)X_1^- - \frac{x_1}{x_2}\left(1+\frac{h}{x_2+a}\right)X_2^-\right], \tag{6.14}$$

and equation (5.31) gives

$$\hat{x}_1\hat{x}_2 = -\frac{1}{a\nu}(x_1x_2+\phi_1)v_2^2w + \frac{1}{a}x_1x_2v_2w + \frac{h}{a\nu}(x_1x_2+\pi_1)v_2$$
$$- \frac{h}{a\nu^2}[2x_1x_2 + h(x_1+x_2) + h^2 - 3a^2$$
$$- (x_1+x_2+2h)(x_1+x_2+2h+3a) + \pi_1 + \phi_1 + \pi_2\phi_2]v_2^2. \tag{6.15}$$

So, with equations (6.5) and (6.10), this gives $\hat{x}_1\hat{x}_2$ in terms of $\{x_i, X_i^{\pm}\}$. Therefore we have reconstructed $\hat{w}$ in terms of $\{x_i, X_i^{\pm}\}$. With the reconstruction of v_4 this then gives a reconstruction of v_3, and from the Casimir (3.3), $v_1 + v_3 + v_5 = \nu$, this gives a reconstruction of v_5. Therefore, full reconstruction has been achieved.

7. Conclusion

The quantum discrete Dubrovin equations have been derived. Equations (5.14) and (5.20) give a temporal evolution and (5.29) and (5.31) give a spatial evolution.

The classical discrete Dubrovin equations were published in [17, 22], but in [17] the focus was on temporal equations only. In the temporal case, the work presented here is very much analogous to the classical case. The classical ($h \to 0$) limit of (5.13) is the discrete Dubrovin equations as presented in [17]. This is seen as follows. The derivation of the (classical) discrete Dubrovin equations employs the invariant spectral curve

$$\det(T(\lambda) - \eta) = 0, \tag{7.1}$$

which defines a hyperelliptic curve of genus $g = P - 1$. The classical equations are written in terms of the discriminant of the hyperelliptic curve (7.1), $R(\lambda)$. This discriminant may be expressed as

$$R(\lambda) = \tfrac{1}{4}(A(\lambda) + D(\lambda))^2 - \det(T(\lambda)). \tag{7.2}$$

Following [17], we see that (7.2) implies that

$$\tfrac{1}{2}(A(\lambda) - D(\lambda)) = \kappa\sqrt{R(\lambda) - B(\lambda)C(\lambda)},$$

where κ denotes the sign $\pm$ and corresponds to the choice of sheet of the Riemann surface, the condition $\tilde{\kappa} = -\kappa$ can be seen by also considering (3.8) and evaluating at the (no longer *operator*) roots of $B(\lambda)$. A quantum deformation of the expression (7.2), evaluated at an operator root of $B(\lambda)$, x_n, is

$$R(x_n) = \frac{1}{4}\tau(x_n)^2 - \frac{1}{2}\left(\Delta\left(x_n - \frac{h}{2}\right) + \Delta\left(x_n + \frac{h}{2}\right)\right) = \frac{1}{4}\left(X_n^- - X_n^+\right)^2. \tag{7.3}$$

Therefore, in the classical limit, the quantum discrete Dubrovin equations, (5.13), become

$$\widetilde{\mathcal{M}}^{-1}\left(\tilde{\kappa}\sqrt{\mathbf{R}(\tilde{\mathbf{x}}_1)} - \frac{1}{2}I_0\,\mathbf{e}\right) = \mathcal{M}^{-1}\left(-\kappa\sqrt{\mathbf{R}(\mathbf{x}_1)} - \frac{1}{2}I_0\,\mathbf{e}\right), \tag{7.4}$$

where $\sqrt{\mathbf{R}(\mathbf{x}_1)} = (\sqrt{R(x_1)}, \ldots, \sqrt{R(x_{P-1})})^t$. Equation (7.4) is the classical discrete Dubrovin equation, as first presented in [17]. (The notation of [17] is such that the roots of $B(\lambda)$ are denoted by $\{\mu_i\}$, rather than by $\{x_i\}$.) Classically, the discrete Dubrovin equations are the intermediate step towards the parametrization of the orbits of the classical map. In [17], the parametrization of the solutions of (7.4) in terms of Abelian functions of Kleinian type was discussed, and illustrated in the $P = 2$ and $P = 3$ cases. From the new perspective of this paper it is surprising that the classical limit of (5.20) is not required for this parametrization, and, indeed, does not feature in [17].

Returning to the quantum equations, the next issues to be addressed concern the representation theory. Then, drawing inspiration from the classical discrete Dubrovin equations, we would hope to be able to construct explicit expressions for the quantum propagators interpolating over an arbitrary number of discrete-time steps. Effective mechanisms for computing expectation values and long-time asymptotics for the transition amplitudes would also be desirable corollaries of this proposed work. Extensions of the present work to the higher rank case associated with quantum mappings in the Gel'fand–Dikii hierarcy [18, 19] also form the subject of future work.

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Appendix A. Quantum invariants

Consider the following evolution of the monodromy matrix:

$$T' = NTN^{-1} \tag{A.1}$$

(so, to be specific, N could be M_1 or L_1). In this appendix it will be proved that if N satisfies the relations

$$R^+_{12}N_1N_2 = N_2N_1R^-_{12} \tag{A.2}$$

and

$$T_1N_1^{-1}S^+_{12}N_2 = N_2S^-_{12}T_1N_1^{-1} \tag{A.3}$$

then the Yang–Baxter relation (2.4) is preserved through this evolution and, moreover,

$$\tau(\lambda) = \mathrm{tr}(K(\lambda)T(\lambda))$$

is invariant under this evolution, for a certain $K(\lambda)$ which will also be derived. First the preservation of the Yang–Baxter relation (2.4) under the evolution (A.1) is shown. It requires only (A.2), (A.3), and the constraints on the R and S matrices that $R^\pm_{12}S^\pm_{12} = S^\mp_{12}R^\mp_{12}$,

$$\begin{aligned}
R^+_{12}T'_1S^+_{12}T'_2 &= R^+_{12}N_1T_1N_1^{-1}S^+_{12}N_2T_2N_2^{-1}\\
&= R^+_{12}N_1N_2S^-_{12}T_1N_1^{-1}T_2N_2^{-1}\\
&= N_2N_1S^+_{12}R^+_{12}T_1N_1^{-1}T_2N_2^{-1}\\
&= N_2N_1S^+_{12}R^+_{12}T_1S^+_{12}T_2N_2^{-1}N_1^{-1}S^{-^{-1}}_{12}\\
&= N_2N_1S^-_{21}T_2S^-_{12}T_1R^-_{12}N_2^{-1}N_1^{-1}S^{-^{-1}}_{12}\\
&= T'_2S^+_{21}N_1N_2S^-_{12}T_1N_1^{-1}N_2^{-1}S^{+^{-1}}_{12}R^-_{12}\\
&= T'_2S^-_{12}T'_1R^-_{12}.
\end{aligned} \tag{A.4}$$

The derivation of the invariants of the evolution (A.1) will now be given. Introduce the tensor

$$K_{12} = P_{12}K_1K_2, \tag{A.5}$$

where P_{12} is the permutation operator, which satisfies the relations

$$P_{12}(A\otimes B) = (B\otimes A)P_{12} \qquad P_{12} = P_{21} \qquad \mathrm{tr}_2\, P_{12} = \mathbf{1}_1. \tag{A.6}$$

Choosing $\lambda_1 = \lambda_2$, we can take the trace of (A.3) contracted with K_{12}. The left-hand side leads to

$$\mathrm{tr}_1\,\mathrm{tr}_2\big(K_{12}T_1N_1^{-1}S^+_{12}N_2\big) = \mathrm{tr}_2\big(K_2T_2N_2^{-1}\,\mathrm{tr}_1\big(P_{12}K_2S^+_{12}\big)N_2\big) = \mathrm{tr}(KT) \tag{A.7}$$

provided that

$$\mathrm{tr}_1\big(P_{12}K_2S^+_{12}\big) = \mathbf{1}_2. \tag{A.8}$$

Under the same condition, (A.8), we have, from the right-hand side of equation (A.3), that

$$\mathrm{tr}_1\mathrm{tr}_2\left(K_{12}N_2S^-_{12}T_1N_1^{-1}\right) = \mathrm{tr}_1\big(K_1N_1\,\mathrm{tr}_2(P_{12}K_1S^-_{12})T_1N_1^{-1}\big) = \mathrm{tr}(KT'). \tag{A.9}$$

Hence, if $K(\lambda)$ is a solution of equation (A.8), we have the invariance of $\mathrm{tr}(K(\lambda)T(\lambda))$ under the evolution given by equation (A.1).

A solution to equation (A.8) is found by taking

$$K_2 = \mathrm{tr}_1\left\{P_{12}^{t_1}\left[\left({}^{t_1}S_{12}^{+}\right)^{-1}\right]\right\}. \tag{A.10}$$

This is most easily verified by introducing the twisted product

$$X_{12} * Y_{12} :=^{t_2} \left({}^{t_2}X_{12}\,{}^{t_2}Y_{12}\right) =^{t_1} \left({}^{t_1}Y_{12}\,{}^{t_1}X_{12}\right).$$

An inverse with respect to the * product is

$$X_{12}^{-1*} =^{t_2} \left(\left({}^{t_2}X_{12}\right)^{-1}\right) =^{t_1} \left(\left({}^{t_1}X_{12}\right)^{-1}\right).$$

So, with equation (A.10),

$$\begin{aligned}
\mathrm{tr}_1\left(P_{12}K_2S_{12}^{+}\right) &= \mathrm{tr}_1\,\mathrm{tr}_3\left(P_{12}P_{32}\left(S_{32}^{+}\right)^{-1*}S_{12}^{+}\right)\\
&= \mathrm{tr}_1\,\mathrm{tr}_3\left(P_{32}P_{13}\left(S_{32}^{+}\right)^{-1*}S_{12}^{+}\right)\\
&= \mathrm{tr}_3\left(P_{32}\,\mathrm{tr}_1\left(P_{13}\left(S_{32}^{+}\right)^{-1*}S_{12}^{+}\right)\right)\\
&= \mathrm{tr}_3\left(P_{32}S_{32}^{+} * \left(S_{32}^{+}\right)^{-1*}\right)\\
&= \mathrm{tr}_3(P_{32}\mathbf{1}_{32}) = \mathbf{1}_2.
\end{aligned}$$

For the proof of the commutativity of the invariants $\tau(\lambda) = \mathrm{tr}(K(\lambda)T(\lambda))$ with the form of $K(\lambda)$ given in equation (A.10) we refer the reader to [20].

Appendix B. Quantum determinant factorization

In the case of ultralocal models it is straightforward to show that the quantum determinant is equal to the product of the local quantum determinants of the constituent L operators. This factorization will now be proved for the present non-ultralocal case, which is less straightforward.

Equations (2.1a) to (2.1c) and (2.7) give

$$R_{12}^{+}\overleftarrow{\prod_{j=1}^{m}}L_{j,1}\overleftarrow{\prod_{k=1}^{m}}L_{k,2} = \overleftarrow{\prod_{j=1}^{m}}L_{j,2}\overleftarrow{\prod_{k=1}^{m}}L_{k,1}R_{12}^{-}, \tag{B.1}$$

for $m \leqslant P-1$. From equations (B.1) and (2.7),

$$R_{12}^{-}S_{12}^{-}\overleftarrow{\prod_{j=1}^{m}}L_{j,1}\overleftarrow{\prod_{k=1}^{m}}L_{k,2} = S_{12}^{+}\overleftarrow{\prod_{j=1}^{m}}L_{j,2}\overleftarrow{\prod_{k=1}^{m}}L_{k,1}R_{12}^{-}. \tag{B.2}$$

Therefore, for the particular relative value of the spectral parameters λ_1 and λ_2 such that R_{12}^{-} is proportional to a rank-one projector, equation (B.2) is defined to be equal to $R_{12}^{-}\mathrm{Qet}\big(\overleftarrow{\prod}_{j=1}^{m} L_j\big)$.

Consider the left-hand side of equation (B.2),

$$\begin{aligned} R^-_{12}\mathrm{Qet}\left(\overleftarrow{\prod_{j=1}^{m}} L_j\right) &= R^-_{12}S^-_{12}\overleftarrow{\prod_{j=1}^{m}} L_{j,1}\overleftarrow{\prod_{k=1}^{m}} L_{k,2} \\ &= R^-_{12}S^-_{12}L_{m,1}L_{m,2}S^+_{21}\overleftarrow{\prod_{j=1}^{m-1}} L_{j,1}\overleftarrow{\prod_{k=1}^{m-1}} L_{k,2} \\ &= S^+_{12}L_{m,1}L_{m,2}R^-_{12}S^-_{12}\overleftarrow{\prod_{j=1}^{m-1}} L_{j,1}\overleftarrow{\prod_{k=1}^{m-1}} L_{k,2} \\ &= \mathrm{Qet}(L_m)R^-_{12}\mathrm{Qet}\left(\overleftarrow{\prod_{j=1}^{m-1}} L_{j,1}\right), \end{aligned} \tag{B.3}$$

and, therefore,

$$\mathrm{Qet}\left(\overleftarrow{\prod_{j=1}^{m}} L_j\right) = \overleftarrow{\prod_{j=1}^{m}}\mathrm{Qet}(L_j), \tag{B.4}$$

where $m \leqslant P-1$. The proof of the same result via the right-hand side of (B.2) follows similarly.

The quantum determinant for the Yang–Baxter structure (2.4), Δ, was introduced in equation (2.16). From the left-hand side of (2.15),

$$\begin{aligned} R^-_{12}\Delta &= R^-_{12}S^-_{12}T_1S^+_{12}T_2 \\ &= R^-_{12}S^-_{12}L_{P,1}L_{P,2}S^+_{21}\overleftarrow{\prod_{j=1}^{P-1}} L_{j,1}\overleftarrow{\prod_{k=1}^{P-1}} L_{k,2} \\ &= S^+_{12}L_{P,2}L_{P,1}R^-_{12}S^-_{12}\overleftarrow{\prod_{j=1}^{P-1}} L_{j,1}\overleftarrow{\prod_{k=1}^{P-1}} L_{k,2}, \end{aligned} \tag{B.5}$$

(the same result follows similarly for the right-hand side) and hence, from equation (B.4), we have

$$\Delta = \overleftarrow{\prod_{j=1}^{P}}\mathrm{Qet}(L_j).$$

Appendix C. B_{P-1} and C_P are in the centre of the algebra

It can be shown, using the commutation relations (2.4) with the explicit realization of the R and S matrices (3.10) and the gradation of the monodromy matrix (3.6), that B_{P-1} and C_P belong to the centre of the algebra. The model specific proof given here, however, is more constructive for our purposes as it reveals B_{P-1} and C_P to have the value prescribed to the Casimirs in equation (3.3). Consider L_{P+1},

$$L_{P+1} = \begin{pmatrix} \lambda + a + v_{2P+2}v_{2P+1} & v_{2P+2} \\ \lambda v_{2P+1} & \lambda \end{pmatrix}. \tag{C.1}$$

From the definition of the monodromy matrix, equation (3.5), and the grading of the monodromy matrix for period P, equation (3.6), along with equation (C.1), we observe that the monodromy matrix of period $P+1$, T_{P+1}, has the form

$$\begin{pmatrix} \lambda^{P+1} + \lambda^{P}[A_{P-1} + a + v_{2P+2}v_{2P+1} + v_{2P+2}C_P] + \cdots & \lambda^{P}[B_{P-1} + v_{2P+2}] + \cdots \\ \lambda^{P+1}[C_P + v_{2P+1}] + \cdots & \lambda^{P+1} + \lambda^{P}[D_{P-1} + v_{2P+1}B_{P-1}] + \cdots \end{pmatrix}.$$

The B and C entries give us recursion relations for the coefficients of the highest order of λ. Using these, in conjunction with the information from (C.1) that for the period $= 1$ case $C_P = v_1$ and $B_{P-1} = v_2$, we obtain that for the period $= P$ case

$$B_{P-1} = \sum_{n=1}^{P} v_{2n} \qquad C_P = \sum_{n=1}^{P} v_{2n-1}. \tag{C.2}$$

For the mappings of KdV type these summations are the Casimirs given in equation (3.3).

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